Lecture 5: Proof of Nash's Theorem

Wednesday, 7 September 2022 9:45 AM

Theorem (Nash, 1950): Every game w/ a finite set of players & strategies has an equilibrium in mixed strategies

- proves existence, some properties, but not constructive
- uses "fixed-pt. theorems"

Theorem (Brouwer's Fixed-Pt. Theorem): Let $X \subset \mathbb{R}^n$ be convex and compact. Let $\varphi: X \to X$ be a continuous function. Then φ has a fixed point in X, i.e., $\exists x \in X$ s.t. $\varphi(x) = x$.

Intuitively, let $\Gamma = (N, \{S_i\}_{i\in N}, \{u_i\}_{i\in N})$ be an n-person game.
Suppose each player has m pure strategies.
Then for each player i, $\Sigma_i = \Delta_m = \{x \in \mathbb{R}^m : \sum_j x_j = 1\} \subseteq \mathbb{R}^m$
& $\Sigma = \Sigma_1 \times \dots \times \Sigma_n \subseteq \mathbb{R}^{m\times n}$
Consider a function $\varphi: \Sigma \to \Sigma$, defined as:

$$\varphi(\sigma_1, \sigma_2, \dots, \sigma_n) = (\sigma_1', \sigma_2', \dots, \sigma_n')$$

where, if $\sigma = (\sigma_1, \dots, \sigma_n)$,
σ_i' is the best-response to σ_{-i}
i.e., $\sigma_i' = \arg\max_{x\in\Sigma_i} u_i(x, \sigma_{-i})$

Then, if σ is a fixed-pt. for φ,
i.e., $\varphi(\sigma_1, \sigma_2, \dots, \sigma_n) = (\sigma_1, \dots, \sigma_n)$
then σ is a NE.
(Problem: note that there can be infinite best-responses, so φ is not well-defined)

So let's do this more formally.

(i) $X \subseteq \mathbb{R}^n$ is <u>convex</u> if, $x_1, x_2 \in X \Rightarrow \lambda x_1 + (1-\lambda)x_2 \in X$ $\forall \lambda \in [0,1]$

(ii) $X \subseteq \mathbb{R}^n$ is <u>bounded</u> if $\exists M \in \mathbb{R}$ s.t. $X \subseteq B(0,M) = \{x \in \mathbb{R}^n : \|x\|_2 \leq M\}$

(iii) $X \subseteq \mathbb{R}^n$ is <u>closed</u> if it contains all its limit points:
(a) If $x_1, x_2, \dots$ is a convergent sequence, $x_i \in X$, and $\lim_{i\to\infty} x_i = \hat{x}$, then $\hat{x} \in X$.
X is <u>open</u> if $\forall x \in X$, $\exists \varepsilon > 0$ s.t. $B(x,\varepsilon) = \{y \in \mathbb{R}^n : \|x-y\|_2 \leq \varepsilon\} \subseteq X$
(b) X is <u>closed</u> if $\mathbb{R}^n \setminus X$ is open

(iv) f is continuous ((ε,δ)-continuity) if $\forall x \in X, \forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall y \in X: \|y-x\|_2 \leq \delta$, $\|f(y) - f(x)\|_2 \leq \varepsilon$

Examples & Counterexamples

$X \subseteq \mathbb{R}$:
1. not bounded: $X = \{x \in \mathbb{R} : x \geq 0\}$
2. not closed: $X = [0,1)$
3. not convex: $X = [0,1] \cup [2,3]$

BFPT in 1-dimension:
If X is convex & compact, then $X = [a,b]$
Let f be a continuous fn. If $f(a) \neq a$, $f(b) \neq b$,
then $f(a) > a$, $f(b) < b$
$\Rightarrow f(a) - a > 0$, $f(b) - b < 0$
Let $g(x) = f(x) - x$, $x \in [a,b]$. Then since $g(a) > 0$, $g(b) < 0$, by intermediate value theorem, $\exists x \in [a,b]$ s.t. $0 = g(x) = f(x) - x$. This is a fixed pt.

Examples of no fixed-pt.:
- f continuous, X not bounded:
  $X = [0,\infty)$, $f(x) = 2x$
- f continuous, X not closed:
  $X = (0,1]$, $f(x) = x/2$
- f continuous, X not convex:
  $X = [0,1] \cup [2,3]$, $f(x) = \begin{cases} x+2 & \text{if } x \leq 1 \\ x-2 & \text{if } x \geq 2 \end{cases}$
- f discontinuous, X compact & convex:
  $X = [0,1]$, $f(x) = \begin{cases} x + 1/2 & x < 1/2 \\ x - 1/2 & x \geq 1/2 \end{cases}$

Proof of Nash's Theorem via BFPT

(Note: Can prove Nash's Theorem using other fixed-pt. theorems, eg. Kakutani Fixed-Pt. Theorem. Will stick to BFPT for now)

n-player game $(N, \{S_i\}_{i\in N}, \{u_i\}_{i\in N})$
For simplicity, assume each player has m pure strategies
Recall: $\Sigma_i = \Delta_m \subseteq \mathbb{R}^m$, $\sigma_i \in \Sigma_i$
$\Sigma = \Sigma_1 \times \dots \times \Sigma_n$, $\sigma \in \Sigma \subseteq \mathbb{R}^{m\times n}$
Note: Σ is compact & convex (prove yourself)
Given σ, σ_{-i}, $u_i(\sigma)$, ...
Defn: $\theta_i(x, \sigma) = u_i(x, \sigma_{-i}) - \|x - \sigma_i\|_2^2$ $(x \in \Sigma_i)$
"distance-adjusted utility"
Defn: $\varphi_i(\sigma) = \arg\max_{x\in\Sigma_i} \left[u_i(x, \sigma_{-i}) - \|x - \sigma_i\|_2^2\right]$

i.e., $\varphi_i(\sigma)$ is <u>the</u> mixed strategy for i that maximizes the distance-adjusted utility.
(will show later that $\varphi_i(\sigma)$ is well-defined)
$\varphi(\sigma) = (\varphi_1(\sigma), \varphi_2(\sigma), \dots, \varphi_n(\sigma))$

Need to show:
(i) $\varphi_i(\sigma)$ is well-defined
(ii) φ, Σ satisfy conditions for BFPT
(iii) fixed-pt. for φ is a NE for the game.

For (i), will show that "distance-adjusted utility" $\theta_i(x,\sigma)$ is strictly concave in x (keeping σ fixed), and hence $\theta_i(\cdot,\sigma)$ has a unique maximizer (and hence $\varphi_i(\sigma)$ is well-defined)

Lemma: $\forall i \in N$, $\forall \sigma \in \Sigma$, $\theta_i(x,\sigma)$ is strictly concave in the first argument.
Since $\theta_i(x,\sigma) = u_i(x, \sigma_{-i}) - \|x - \sigma_i\|_2^2$, let's consider these individually.
Claim: The function $u_i(x, \sigma_{-i})$ is linear in the first argument
Proof: Let $u_i(s_k, \sigma_{-i})$ be the expected utility for the k^{th} pure strategy, given σ_{-i}. These are fixed, given σ_{-i}
Then $u_i(x, \sigma_{-i}) = \sum_{k=1}^{m} x_k u_i(s_k, \sigma_{-i})$

Hence linear. ∎

Also, $\|x - \sigma_i\|_2^2 = \sum_{k=1}^{m} (x_k - \sigma_i(k))^2$ is strictly convex
Hence, $\theta_i(x,\sigma) = u_i(x, \sigma_{-i}) - \|x - \sigma_i\|_2^2$ is strictly concave.
Since a strictly concave fn. on a convex set has a unique maximizer, $\varphi_i(\sigma) = \arg\max_{x\in\Sigma_i} \theta_i(x,\sigma)$ is well-defined.

For (ii), clearly Σ is compact & convex.
Need to show that $\varphi(\sigma)$ is continuous $\Leftrightarrow$ $\varphi_i(\sigma)$ is continuous for all $i \in N$.

Maximum Theorem: Let $X, Y \subseteq \mathbb{R}^n$ be compact.
$f: X \times Y \to \mathbb{R}$ is continuous & strictly concave in X, and $\varphi: Y \to X$ be defined as $\varphi(y) = \arg\max_{x\in X} f(x,y)$
Then $\varphi(\cdot)$ is continuous.

To show $\varphi_i(\sigma)$ is continuous, note that
$\varphi_i(\sigma) = \arg\max_{x\in\Sigma_i} \theta_i(x,\sigma)$
we've shown $\theta_i(x,\sigma)$ is continuous & strictly concave in the first argument, Σ_i, Σ are compact
Hence by maximum theorem, $\varphi(\sigma)$ is continuous.

Thus by BFPT, $\varphi: \Sigma \to \Sigma$ has fixed-pt.

Lemma: Let $\varphi(\hat{\sigma}) = \hat{\sigma}$ for $\sigma \in \Sigma$. Then $\hat{\sigma}$ is a NE.
Proof: $\hat{\sigma} = (\hat{\sigma}_1, \dots, \hat{\sigma}_n)$. & $\hat{\sigma}_i$ uniquely maximizes $\theta_i(x, \hat{\sigma}) = u_i(x, \hat{\sigma}_{-i}) - \|x - \hat{\sigma}_i\|_2^2$
Say $\hat{\sigma}$ is not a NE. Thus $\exists i$, $\sigma_i' \in \Sigma_i$ s.t.
$u_i(\sigma_i', \hat{\sigma}_{-i}) > u_i(\hat{\sigma}_i, \hat{\sigma}_{-i})$
Let $\sigma_i^\lambda = \lambda\sigma_i' + (1-\lambda)\hat{\sigma}_i$
WTS: $\exists \lambda \in (0,1]$ s.t. $\theta_i(\sigma_i^\lambda, \hat{\sigma}) > \theta_i(\hat{\sigma}_i, \hat{\sigma})$ for a contradiction.
Now, $\theta_i(\sigma_i^\lambda, \hat{\sigma}) - \theta_i(\hat{\sigma}_i, \hat{\sigma})$

$$= u_i(\sigma_i^\lambda, \hat{\sigma}_{-i}) - u_i(\hat{\sigma}_i, \hat{\sigma}_{-i}) - \left(\|\sigma_i^\lambda - \hat{\sigma}_i\|_2^2\right)$$

(skipping some calculations)

$$= \lambda\left(u_i(\sigma_i', \hat{\sigma}_{-i}) - u_i(\hat{\sigma}_i, \hat{\sigma}_{-i})\right) - \lambda^2\|\sigma_i' - \hat{\sigma}_i\|_2^2$$

for $\lambda \in \left(0, \frac{u_i(\sigma_i', \hat{\sigma}_i)}{\|\sigma_i' - \hat{\sigma}_i\|_2^2}\right)$, this is > 0

hence, $\exists \lambda > 0$ s.t. $\theta_i(\sigma_i^\lambda, \hat{\sigma}) > \theta_i(\hat{\sigma}_i, \hat{\sigma})$
& hence $\hat{\sigma}_i$ is not maximizer of $\theta_i(\cdot, \hat{\sigma})$, giving a contradiction. ∎

This completes proof of Nash Theorem.